Solid of Revolution

IB Mathematic HL

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Introduction:

Geometry is one of the areas of math that I feel passionate about, thereby I am inspired to do my Internal Assessment in this area. I found that ellipsoid is an interesting geometric shape. It is not as perfectly round as a sphere, and it has two or three distinct radii, which peaks my attention and interest. I have learned many geometric shapes, however, ellipse and ellipsoid are the two I had very little knowledge about. We see ellipsoid in our daily lives, such as potatoes, watermelons and stones. In fact, our Earth is an ellipsoid as well. Besides finding out the volume of an ellipsoid, I wondered if we can use volume of revolution to find the volume of other shapes as well. Therefore, it inspires me to do this paper on finding the volume of an ellipsoid and the volume under a curve by using the disk method and the cylinder method. An ellipsoid is an ellipse in 3D. It has three axials, and three semi-axes. There are two kinds of ellipsoid. One has three distinct semi-axes, and the other one has a pair of equivalent semi-axes, and one distinct semi-axes. I am going to explore ellipsoids with a pair of equivalent and one distinct semi-axes.

I have never heard or learnt about the volume of revolution before I did this Internal Assessment. My math teacher mentioned this process, which brought my interest to investigate this topic. Solid of revolution, it is also called the volume of revolution, which includes the disk method and cylinder method. It is a solid figure that can be constructed by rotating a plane line around an axis, which creates a solid in a 3D shape. Essentially, allowing us to calculate the volume of a geometric shape in 3D between two points in a curve using integration. Integration is something that we learned early in grade 11, however, being able to apply this knowledge to the volume of revolution is very interesting, showing that areas of math are connected. Solid of revolution allows us to find volumes of many irregular geometric shape, which made me curious.
how do I find the volume of any irregular geometric shapes, such as a Torricelli Trumpet, a ring and etc.

Investigation:

To find the volume of an ellipsoid in 3D, first of all, is to find the equation of an ellipse corresponding to the ellipsoid. Figure 1 is an example of an ellipse.

Figure 1

Considering each square is 1 cm X 1 cm, the ellipse shown above has a 4 cm of major axis and 2 cm of minor axis. A major axis is the longest diameter in an ellipsoid, and a minor axis is the shortest diameter in an ellipsoid. The general form of the equation of an ellipse is

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,
\]

where \((h, k)\) is the point of the center of the ellipse, \(a\) is the semi-major axis, which is half of the length of a major axis, and \(b\) is the minor axis, which is half of the length of
a minor axis. In this case, the center of the ellipse is (0, 0), therefore h and k are 0, α is 4, and b is 2, which means that the equation for this ellipse is:

\[
\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1
\]

Simplifying this equation into:

\[
x^2 + 4y^2 = 16
\]

Then putting y by itself on one side of the equation, which is

\[
y = \pm \sqrt{4 - \frac{x^2}{4}}
\]

\[
y = +\sqrt{4 - \frac{x^2}{4}} \text{ is the equation for the top half of the ellipse on the graph, cut by the x axis.}
\]

\[
y = -\sqrt{4 - \frac{x^2}{4}} \text{ is the equation for the bottom half of the ellipse on the graph, cut by the x axis.}
\]

If we were to find the area under the curve, which is the top half of the ellipse with the

equation \( y = +\sqrt{4 - \frac{x^2}{4}} \). Imagine cutting this half sized ellipse into very tiny rectangular strips, and adding their areas together, which it would be the area of the half sized ellipse. Every tiny rectangle has a height of y, and width of dx, as shown in Figure 2. Then, we rotate the curve for 360 degrees around the x axis, as Figure 2 shown.

Figure 2
It would not matter which half of the ellipse we chose to rotate, both of the curves will form the same ellipsoid after rotating around the x axis. After rotating the curve for 360 degrees, as Figure 3 shown, an ellipsoid is formed. Two semi-axes have the same length of 2cm, and the third semi-axes has a length of 4cm. The rectangle that is described above, is being rotated around the x axis for 360 degrees as well, forming a disk, as Figure 4 shown.

The same idea of finding the area of an ellipse, we can sum up the volume of all of the tiny disks to find the volume of the ellipsoid. If we magnified one of the disks, as shown in Figure 4, it is a side way view of a cylinder. The formula of finding the volume of a disk is base area times the height. In this case, the radius of a disk is \( y \), which is determined by the equation, 

\[
y = \sqrt{4 - \frac{x^2}{4}}
\]

and its width is \( dx \). Therefore, the base area of a disk, which is the same as finding the base area of a circle, is \( \pi y^2 \). To find the volume, we need to multiply \( \pi y^2 \) and \( dx \) together, which then we get \( \pi y^2 dx \). Then, substitute the equation \( y = \sqrt{4 - \frac{x^2}{4}} \) into \( \pi y^2 dx \), which equals to
\[
\pi \left( \sqrt{4 - \frac{x^2}{4}} \right)^2 \, dx. \text{ The last step is to add up the volume of each continuous slice of disks in the ellipsoid to find the volume of the ellipse, which is the purpose of integral. The general form of integral is } \int_a^b f(x) \, dx.
\]

The boundary of area of the volume is between \( x = -4 \) and \( x = 4 \), and \( f(x) \) is the equation for the curve that is being rotated, which is \( \sqrt{4 - \frac{x^2}{4}} \), therefore, the volume of the ellipsoid is equal to

\[
\int_{-4}^{4} \pi \left( \sqrt{4 - \frac{x^2}{4}} \right)^2 \, dx.
\]

We can take \( \pi \) to the front of the expression, because it is a constant, it will not affect the calculation:

\[
= \pi \int_{-4}^{4} 4 - \frac{x^2}{4} \, dx
\]

\[
= \pi \left[ 4x - \frac{x^3}{12} \right]_{-4}^{4} \, dx
\]

\[
= \pi \left\{ 4(4) - \frac{4^3}{12} - \left[ 4(-4) - \frac{(-4)^3}{12} \right] \right\}
\]

\[
= \pi \left( 16 - \frac{64}{12} + 16 - \frac{64}{12} \right)
\]

\[
= \frac{64}{3} \pi \text{ cm}^3
\]

In conclusion, I have found the volume of the ellipsoid to be \( \frac{64}{3} \pi \text{ cm}^3 \).
If we were using the volume formula of an ellipsoid to solve the problem, which is known as
\[ V = \frac{4}{3} \pi abc, \]
we would get \[ V = \frac{4}{3} \pi (2)(2)(4) = \frac{64}{3} \pi \text{ cm}^3, \]
which is the same result that I found using the disk method.

We looked at the scenario where the ellipse is rotated around the x-axis. Next, we will look at a scenario where ellipse is rotated around y-axis, as Figure 5 shown. The semi-major axis and the semi-minor axis are the same as the first scenario, which is 4cm and 2cm respectively. Therefore, the equation for this ellipse is \[ \frac{x^2}{2^2} + \frac{y^2}{4^2} = 1. \] We can simplify the equation to \[ 4x^2 + y^2 = 16. \]
Because the ellipse is rotated around the y-axis, therefore, its height and radius will be in term of y. It also means that we need to rearrange our equation to express in term of y, which it would be \[ x = \pm \sqrt{4 - \frac{y^2}{4}}. \] As in the first scenario mentioned, it does not matter which equation we use to rotate around. Imagine that there are disks in the ellipsoid, the same as the first scenario. The only thing that is different is that the disks are rotated around the y-axis instead of the x-axis, as Figure 6 shown.
The height of the disk is dy and the radius of the disk is the function y, which then we get the volume of a disk, \( V = \pi \left( \sqrt{4 - \frac{x^2}{4}} \right)^2 dy \). To calculate the volume of the ellipsoid, we can add up all the disks in the ellipsoid, which involves integration. The boundary between the area of volume is between \( y = 4 \) and \( y = -4 \). Therefore, the volume of this ellipsoid is \( \int_{-4}^{4} \pi \left( \sqrt{4 - \frac{y^2}{4}} \right)^2 dy \).

\[
\begin{align*}
= \pi & \int_{-4}^{4} 4 - \frac{y^2}{4} \, dy \\
= & \pi \left[ 4x - \frac{y^3}{12} \right]_{-4}^{4} \\
= & \pi \left( 16 - \frac{64}{12} + 16 - \frac{64}{12} \right) \\
= & \frac{64}{3} \pi \text{ cm}^3
\end{align*}
\]
The volume of the ellipse in two different scenarios are the same, because the major and minor axis are the same in both scenarios. However, one is rotated around the x-axis, the other one is rotated around the y-axis, which we use the same approach, yet different expression of the equation. The disk method can be used in other shapes besides ellipsoid, for example, a Torricelli Trumpet, volume under a wave function after rotating around axis and etc.

After understanding the basic idea of volume of revolution in rotating around both axis using disk method. Figure 7 is a more difficult scenario that is rotated around the y-axis as well, but using a different method, which is called the cylinder method, and also known as the shell method. It can be solved using the disk method, however, it is much more difficult and there are many areas to make mistakes than the cylinder method. In figure 7, we are finding the volume under a curve after it has been rotated around the y-axis for 360 degrees. The equation of the curve is $y = \frac{1}{8}x^2 - \frac{1}{2}x + \frac{5}{2}$, and we are only looking at finding the volume under this curve from $x=2$ to $x=4$.

Figure 7
If we spin this section of the curve 360 degrees, we get a shape like a cylinder, with a hollow center and a convex opening at the top, as Figure 8 shown.

Figure 8

Imagine that we have many different sizes and heights of cans without covers and bottoms, starting with a can with radius of 2cm, stacking them perfectly tight together, and at the end, it could form the shape as Figure 8 shown. If we add the area of all of the cans that form the shape, we would find the volume of this particular shape, and this is the idea of the cylinder method.

First of all, we need to know how to find the area of cans. If we cut a can vertically, and flatten it out, we would see a rectangle, as Figure 9 shown, which means that the area of the rectangle is the same as the area of the can. The length of the rectangle is the circumference of the can and the width of the rectangle is the height of the can.
The formula for calculating the area of the rectangle is base times height, which in this case is the circumference of the can times the height of the can. The circumference of a can is calculated by multiplying 2, the radius and π together, which in the case, the radius is x. Therefore the circumference of a can is 2πx. The height of a can is y, which is determined by the equation, $y = \frac{1}{8}x^2 - \frac{1}{2}x + \frac{5}{2}$. Therefore, we get that the area of a can is $2\pi x\left(\frac{1}{8}x^2 - \frac{1}{2}x + \frac{5}{2}\right)$. As we see, in Figure 7, the boundary is between x=2 and x=4. Therefore the volume of this shape is equal to $\int_2^4 2\pi x\left(\frac{1}{8}x^2 - \frac{1}{2}x + \frac{5}{2}\right)dx$.

Because 2π is a constant, we can take it out in front of the integral.

$$= 2\pi \int_2^4 \left(\frac{1}{8}x^3 - \frac{1}{2}x^2 + \frac{5}{2}x\right)dx$$

$$= 2\pi \left[\frac{x^4}{32} - \frac{x^3}{6} + \frac{5x^2}{4}\right]_2^4$$
\[ \begin{align*}
&= 2\pi \left[ \frac{256}{32} - \frac{64}{6} + \frac{80}{4} - \left( \frac{16}{32} - \frac{8}{6} + \frac{20}{4} \right) \right] \\
&= 2\pi \left[ \left( 8 - \frac{32}{3} + 20 \right) - \left( \frac{1}{2} - \frac{4}{3} + 5 \right) \right] \\
&= 2\pi \left( \frac{52}{3} - \frac{25}{6} \right) \\
&= \frac{79\pi}{3} \text{ cm}^3
\end{align*} \]

As the calculation shown, the volume of this particular shape created by the equation, \( y = \frac{1}{8} x^2 - \frac{1}{2} x + \frac{5}{2} \), is \( \frac{79\pi}{3} \text{ cm}^3 \).

Conclusion:

In conclusion, I have learned what the solid of revolution is about, hence finding the volume of an ellipsoid rotated around both axis and the volume of other shapes, such as a Torricelli Trumpet and other irregular shapes created by a curve. I have also learnt that the general form of the volume of an ellipsoid is \( \pi \int_{a}^{b} [f(x)]^2 \, dx \). I was fascinated by the two methods, when I first learned them, which shows me the diversity of methods in math. During the investigation, it was difficult to apply math with the technology, visualize the 3D shapes, understand the concepts of the two methods and be able to explain them. Because I did not learn about this topic in class from the teacher, I noticed the difficulty of self-teaching myself. However, after this investigation, I realized that being able to be self-taught is an important skill to have, not only in math, as well as in life.
Work Cites

